Regularity and Normality in Soft Bitopological Ordered Spaces

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Abstract
This paper examines regularity and normality in soft separation axioms for soft bitopological ordered spaces and their relationships with other properties. The findings expand our understanding of bitopological ordered spaces. Previous research, such as Al-Shami's work [3], has established distinctions between separation axioms in topological ordered spaces, which are more effective in describing these spaces' properties.

Keywords:
Soft topology, soft bitopological ordered space and bi−ordered soft regularity and normality.

1. Introduction

Nachbin [17] (1965) introduced the concept of a topological ordered space by adding a partial order to the structure of a topological space. McCartan [13] (1968) used monotone neighborhoods to study ordered separation axioms in these spaces. To address vagueness and uncertainty in real-life problems, mathematical tools like fuzzy sets, intuitionistic fuzzy sets, rough sets, vague sets, and soft sets have been developed. Molodtsov [16] (1999) introduced soft sets as a mathematical tool to handle vagueness and uncertainty. Maji et al. [15] and [14] (2003, 2002) and Aktas and Cagman [1] (2007) further developed soft set theory and its application in decision making problems and algebraic structures. Shabir and Naz [20] (2011) investigated soft separation axioms for crisp points, and Hussain and Ahmad [7] (2011) examined properties of soft interior, closure, and boundary. Nazmul and Samanta [18] (2013) studied neighborhood properties of soft topological spaces, and four different types of separation axioms were defined and discussed in a series of papers (Gocur [10] 2015, Hussain [8] 2015, Shabir 2011 [20], and Tantawy [22] 2016). Singh and Noorie [21] (2017) extended the understanding of soft topological spaces and their properties. Ittanagi [9] (2014) introduced soft bitopological spaces and soft separation axioms, and Kandil et al. [11] (2016) further studied their structures and defined basic concepts. El-Shafei et al. [3], [4] (2019, 2018) developed two new types of soft relations and introduced the concept of soft topological ordered spaces and ordered soft separation axioms. El-Sheikh et al.[5], [6] (2023, 2022) introduced the concept of soft bitopological ordered spaces and studied the relationships between increasing (decreasing, balancing) pairwise open (closed) soft sets and neighborhoods, and increasing (decreasing) pairwise soft neighbourhoods. This paper discusses the use of soft sets and soft bitopologies in ordered spaces. Section 2 provides definitions and properties of soft sets and soft topologies. Section 3 introduces the concept of"Bi-ordered soft regularity and normality." Examples are provided to demonstrate the connections between these concepts and highlight their characteristics.

2. Preliminaries

References and resources, such as [3], [5], [6], [12], [20], can aid in further exploration and comprehension of specialized mathematical concepts, including soft set theory, soft points, soft topological space, soft topological ordered space, and soft bitopological ordered space. From now on, we will use the notation X to refer to the universe set, E to denote a fixed set of parameters, and 2^X to represent the power set of X.

Definition 2.1 [12] A binary relation $\leq$ on a set X is a partial order relation if it is reflexive, anti-symmetric, and transitive. The equality relation on X, denoted by N, is defined as $\{(a,a) : a \in X\}$.

Definition 2.2 [17] A triple $(X,\tau,\leq)$ is called a topological ordered space when (X,τ) is a topological space and $(X,\leq)$ is a partially ordered set.

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Definition 2.3 [2, 15, 16, 20] A soft set is defined as a pair (M,E), where M : E → 2^X. The notation M_E is used instead of (M,E) for brevity. A soft set can also be represented as a set of ordered pairs, where $M_E = \{(e, M(e))|e \in E, M(e) \in 2^X\}$.

The collection of all soft sets over X is denoted by $P(X)^E$.

A null soft set, denoted by φ, is one where M(e) = ∅ for all e ∈ E. An absolute soft set, denoted by X_E, is one where M(e) = X for all e ∈ E.

Two soft sets, M_E,N_E ∈ $P(X)^E$, are considered a soft subset, denoted by N_E v M_E, if N(e) ⊆ M(e) for all e ∈ E. They are considered equal, denoted by N_E = M_E, if N_E v M_E and M_E v N_E. The union and intersection of two soft sets, N_E and M_E, are represented by N_E t M_E and N_E u M_E, respectively. The difference of two soft sets, N_E and M_E, is denoted by N_E−M_E, and the complement of a soft set N_E is denoted by N^c_E.

Definition 2.4 [18, 19] A soft set N_E : E → 2^X defined as N(α) = a if α = e and N(α) = ∅ if α ∈ E − {e} is called a soft point and denoted by a^e. The collection of all soft points over X is denoted by $Sp(X)^E$. A soft point a^e is said to be belonging to a soft set N_E, denoted by a^e∈bN_E, if for the member e ∈ E, a(e) ⊆ N(e).

Definition 2.5 [3, 16] For a soft set N_E over X and an element a ∈ X, we say a ∈ N_E if a ∈ N(α) for every α ∈ E and a ∈6 N_E if a 6∈ N(α) for some α ∈ E. We say a b N_E if a ∈ N(α) for some α ∈ E and a 6b N_E if a 6∈ N(α) for every α ∈ E. The notations ∈,6∈,b and b6 are respectively read as belong, non-belong, partial belong and total non-belong relations.

Definition 2.6 [20] A soft topology on a set X is a collection of soft sets over X with a certain set of properties. Specifically, the collection must contain the null soft set and the absolute soft set , and it must be closed under arbitrary unions and finite intersections.

A soft topological space is a triple (X,τ,E) where τ is a soft topology on X. The members of τ are called soft open sets, and their complements are called soft closed sets.

Definition 2.7 [18] Let $P(X)^E$ and $P(Y)^K$ be families of soft sets over X and Y , respectively. Let φ : X → Y and ψ : E → K be two mappings. The mapping φ_ψ : $P(X)^E$ → $P(Y)^K$ is a soft mapping from X to Y , denoted by φ_ψ, defined as follows:

1. For G_E ∈ $P(X)^E$, $\varphi_\psi(G_E)(k)$ = S$_{e\in\psi^{-1}(k)}$ G(e) if $\psi^{-1}(k)$ 6= ∅, and $\varphi_\psi(G_E)(k)$ = ∅ otherwise, for all k ∈ K. The soft set $\varphi_\psi(G_E)$ is called the soft image of G_E.

2. For F_K ∈ $P(Y)^K$, $\varphi^{-1}_\psi(F_K)(e) = \varphi^{-1}(F(\psi(e)))$, for all e ∈ E. The soft set $\varphi^{-1}_\psi(F_K)$ is called the soft inverse image of F_K.

Definition 2.8 [23] Let $P(X)^E$ and $P(Y)^K$ be two families of soft sets over X and Y, respectively. A soft mapping φ_ψ : $P(X)^E$ → $P(Y)^K$ is called soft surjective(injective) mapping if φ,ψ are surjective (injective) mappings, respectively. A soft mapping which is a soft surjective and soft injective mapping is called a soft bijection mapping.

Proposition 2.1 [18] Consider φ_ψ : $P(X)^E$ → $P(Y)^K$ is a soft map and let G_E and H_K be two soft subsets of $P(X)^E$ and $P(Y)^K$, respectively. Then we have the following results:

1. $G_E \sqsubseteq \phi_\psi^{-1}(\phi_\psi(G_E))$ and the equality relation holds if φ_ψ is injective.

2. $\varphi_\psi(\varphi^{-1}_\psi(H_K))$ v H_K and the equality relation holds if φ_ψ is surjective.

Definition 2.9 [18] A soft map $\varphi_\psi : (X,\tau,E) \to (Y,\eta,K)$ is said to be:

1. Soft continuous if the inverse image of each soft open subset of (Y,η,K) is a soft open subset of (X,τ,E).
2. Soft open (resp. soft closed) if the image of each soft open (resp. soft closed) subset of (X,τ,E) is a soft open (resp. soft closed) subset of (Y,η,K).
3. Soft homeomorphism if it is bijective, soft continuous and soft open.

Definition 2.10 [9, 11] A quadrable system (X,τ_1,τ_2,E) is called a soft bitopological space when τ_1 and τ_2 are soft topologies on the set X with a fixed set of parameters E. A soft set N_E in a soft bitopological space (X,τ_1,τ_2,E) is called pairwise open soft (PO−soft) if there exists a τ_1−open soft set N_E^1 and a τ_2−open soft set N_E^2 such that $N_E = N_E^1 \sqcup N_E^2$, and pairwise closed soft (PC−soft) if the complement of N_E is a PO−soft set. The family of all PO−soft sets, denoted by τ_{12}, is a supra soft topological space associated with the soft bitopological space (X,τ_1,τ_2,E).

Definition 2.11 [3] A triple $(X,E,\leq)$ is called a partially ordered soft space when $\leq$ is a partial order relation on the set X. An increasing soft operator $i : (P(X)^E,\leq) \to (P(X)^E,\leq)$ and a decreasing soft operator $d : (P(X)^E,\leq) \to (P(X)^E,\leq)$ are defined for each soft set N_E in $P(X)^E$ by $i(N_E)(e) = iN(e) = \{a \in X : b \leq a,$ for some $b \in N(e)\}$ and $d(N_E)(e) = dN(e) = \{a \in X :$ $a \leq b$, for some $b \in N(e)\}$ respectively. A soft subset N_E of the partially ordered soft space $(X,E,\leq)$ is called increasing if $N_E = i(N_E)$, decreasing if $N_E = d(N_E)$.

Proposition 2.2 [3] The following two results hold for a soft map $\varphi_\psi : P(X)^E \to P(Y)^K$.

1. The image of each soft point is soft point.
2. If φ_ψ is bijective, then the inverse image of each soft point is soft point.

Definition 2.12 [3] Let x^e and y^e be two soft points in a partially ordered soft set $(X,E,\leq)$. Then $x^e \leq y^e$ if $x \leq y$.

Definition 2.13 [3] A soft map $\phi_\psi : (P(X)^E, \leq_1) \to (P(Y)^K, \leq_2)$ is said to be:

1. Increasing if $x^e \leq_1 y^e$, then $\varphi_\psi(x^e) \leq_2 \varphi_\psi(y^e)$.
2. Decreasing if $x^e \leq_1 y^e$, then $\varphi_\psi(y^e) \leq_2 \varphi_\psi(x^e)$.
3. Ordered embedding if $x^e \leq_1 y^e$ if and only if $\varphi_\psi(x^e) \leq_2 \varphi_\psi(y^e)$.

Theorem 2.1 [3] The following two results hold for a soft map $\varphi_\psi : (P(X)^E,\leq_1) \to (P(Y)^K,\leq_2)$.

1. If φ_ψ is increasing, then the inverse image of each increasing (resp. decreasing) soft subset of Y_K is an increasing (resp. decreasing) soft subset of X_E.
2. If φ_ψ is decreasing, then the inverse image of each increasing (resp. decreasing) soft subset of Y_K is an decreasing (resp. increasing) soft subset of X_E.

Theorem 2.2 [3] Let $\varphi_\psi : (P(X)^E,\leq_1) \to (P(Y)^K,\leq_2)$ be a bijective ordered embedding soft map. Then the image of each increasing (resp. decreasing) soft subset of X_E is an increasing (resp. decreasing) soft subset of Y_K.

Definition 2.14 [3] A quadrable system $(X,\tau,E,\leq)$ can be rephrased as a soft topological ordered space (STOS) if (X,τ,E) is a soft topological space and $(X,E,\leq)$ is a partially ordered soft space.

Definition 2.15 [3] A soft subset W_E of an STOS $(X,\tau,E,\leq)$ is called increasing (resp. decreasing) soft neighborhood of $x \in X$ if W_E is soft neighborhood of x and increasing (resp. decreasing).

Definition 2.16 [5] The system $(X,\tau_1,\tau_2,E,\leq)$ is said to be a soft bitopological ordered space (SBTOS), if

(X,τ_1,τ_2,E) is a soft bitopological space and (X,E,≤) is a partially ordered soft space.

Definition 2.17 [5] A soft set W_E in a SBTOS (X,τ_1,τ_2,E,≤) is called:

1. ITPS (IPPS)− nbd of a ∈ X if W_E is a total (partial) pairwise soft neighborhood of a and increasing. 2. DTPS (DPPS)− nbd of a ∈ X if W_E is a total (partial) pairwise soft neighborhood of a and decreasing.

Definition 2.18 [6] An SBTOS (X,τ_1,τ_2,E,≤) is said to be:

1. $LPST_1$ $(LPST_1^{**})-$ ordered if for every distinct points x,y in X such that $x \not\lesssim y$ there exists an ITPS (IPPS)− nbd W_E of x such that y 6∈ W_E.
2. $LPST_1^{\bullet}$ $(LPST_1^{*})-$ ordered if for every distinct points x,y in X such that $x \not\lesssim y$ there exists an ITPS (IPPS)− nbd W_E of x such that y 6b W_E.
3. $UPST_1$ $(UPST_1^{**})-$ ordered if for every distinct points x,y in X such that $x \not\lesssim y$ there exists a DTPS (DPPS)− nbd W_E of y such that x 6∈ W_E.
4. $UPST_1^{\bullet}$ $(UPST_1^{*})-$ ordered if for every distinct points x,y in X such that $x \not\lesssim y$ there exists a DTPS (DPPS)− nbd W_E of y such that x 6b W_E.
5. PST_0 $(PST_0^{\bullet})-$ ordered space if it is $LPST_1$ $(LPST_1^{\bullet})-$ ordered or $UPST_1$ $(UPST_1^{\bullet})-$ ordered.
6. PST_0^{*} $(PST_0^{**})-$ ordered space if it is $LPST_1^{*}$ $(LPST_1^{**})-$ ordered or $UPST_1^{*}$ $(UPST_1^{**})-$ ordered.
7. PST_1 $(PST_1^{\bullet})-$ ordered space if it is $LPST_1$ $(LPST_1^{\bullet})-$ ordered and $UPST_1$ $(UPST_1^{\bullet})-$ ordered.
8. PST_1^{*} $(PST_1^{**})-$ ordered space if it is $LPST_1^{*}$ $(LPST_1^{**})-$ ordered and $UPST_1^{*}$ $(UPST_1^{**})-$ ordered.
9. PST_2 $(PST_2^{*})-$ ordered space if for every distinct points x,y in X such that $x \not\lesssim y$ there exist disjoint total (partial) pairwise soft neighborhoods W_E and V_E of x and y, respectively, such that W_E is increasing and V_E is decreasing.
10. $PST_2^{\bullet}$ $(PST_2^{**})-$ ordered space if for every distinct points x,y in X such that $x \not\lesssim y$ there exist disjoint total (partial) pairwise soft neighborhood W_E of x and partial (total) pairwise soft neighborhood V_E of y such that W_E is increasing and V_E is decreasing.

3. Bi−Ordered Soft Regularity and Normality

In this section, a new concept called ”Bi-ordered soft regularity and normality” is introduced and its properties are examined. Various examples are provided to demonstrate the relationships among these concepts and to illustrate the results obtained.

Definition 3.1 For two soft subsets G_E and H_E of an SBTOS (X,τ_1,τ_2,E,≤), we say that G_E is pairwise soft neighborhood of H_E provided that there exists a pairwise open soft set F_E such that H_E v F_E v G_E.

Definition 3.2 An SBTOS (X,τ_1,τ_2,E,≤) is said to be:

1. Lower (resp. upper) PT− soft regularly ordered if for every decreasing (resp. increasing) pairwise closed soft set H_E and x ∈ X such that x 6∈ H_E there exist disjoint pairwise soft neighborhood W_E of H_E and increasing (resp. decreasing) total pairwise soft neighborhood V_E of x such that W_E is decreasing (resp. increasing).
2. Lower (resp. upper) PP− soft regularly ordered if for every decreasing (resp.

increasing) pairwise closed soft set H_E and x ∈ X such that x 6∈ H_E there exist disjoint pairwise soft neighborhood W_E of H_E and increasing (resp. decreasing) partial pairwise soft neighborhood V_E of x such that W_E is decreasing (resp. increasing).

3. Lower (resp. upper) P^*T- soft regularly ordered if for every decreasing (resp. increasing) pairwise closed soft set H_E and x ∈ X such that x 6b H_E there exist disjoint pairwise soft neighborhood W_E of H_E and increasing (resp. decreasing) total pairwise soft neighborhood V_E of x such that W_E is decreasing (resp. increasing).

4. Lower (resp. upper) P^*P- soft regularly ordered if for every decreasing (resp. increasing) pairwise closed soft set H_E and x ∈ X such that x 6b H_E there exist disjoint pairwise soft neighborhood W_E of H_E and increasing (resp. decreasing) partial pairwise soft neighborhood V_E of x such that W_E is decreasing (resp. increasing).

5. Total $P-$ soft regularly ordered if it is both Lower $PT-$ soft regularly ordered and upper $PT-$ soft regularly ordered.

6. Partial $P-$ soft regularly ordered if it is both Lower $PP-$ soft regularly ordered and upper $PP-$ soft regularly ordered.

7. Total P^*- soft regularly ordered if it is both Lower P^*T- soft regularly ordered and upper P^*T- soft regularly ordered.

8. Partial P^*- soft regularly ordered if it is both Lower P^*P- soft regularly ordered and upper P^*P- soft regularly ordered.

9. Lower (resp. upper) $TP-$ soft T_3 ordered if it is both $LPST_1-$ordered (resp. $UPST_1-$ordered) and lower (resp. upper) $PT-$ soft regularly ordered.

10. Lower (resp. upper) $PP-$ soft T_3 ordered if it is both $LPST_1^{**}-$ordered (resp. $UPST_1^{**}-$ordered) and lower (resp. upper) $PP-$ soft regularly ordered.

11. Lower (resp. upper) TP^*- soft T_3 ordered if it is both $LPST_1^{\bullet}-$ordered (resp. $UPST_1^{\bullet}-$ordered) and lower (resp. upper) P^*T- soft regularly ordered.

12. Lower (resp. upper) PP^*- soft T_3 ordered if it is both $LPST_1^{*}-$ordered (resp. $UPST_1^{*}-$ordered) and lower (resp. upper) P^*P- soft regularly ordered.

13. $TP-$ soft T_3 ordered if it is both lower $TP-$ soft T_3 ordered and upper $TP-$ soft T_3 ordered.

14. $PP-$ soft T_3 ordered if it is both lower $PP-$ soft T_3 ordered and upper $PP-$ soft T_3 ordered.

15. TP^*- soft T_3 ordered if it is both lower TP^*- soft T_3 ordered and upper TP^*- soft T_3 ordered.

16. PP^*- soft T_3 ordered if it is both lower PP^*- soft T_3 ordered and upper PP^*- soft T_3 ordered.

Theorem 3.1 An SBTOS $(X,\tau_1,\tau_2,E,\leq)$ is lower (resp. upper) $PT-$soft regularly ordered if and only if for all x ∈ X and every increasing (resp. decreasing) pairwise open soft set U_E containing x, there is an increasing (resp. decreasing) total pairwise soft neighbourhood V_E of x satisfies that $cl^s_{12}(V_E) \sqsubseteq U_E$.

Proof. Necessity: Let x ∈ X and U_E be an increasing pairwise open soft set partially containing x. Then, U_E^c is decreasing pairwise closed soft such that x 6∈ U_E^c . By hypothesis, there exist disjoint decreasing pairwise soft neighbourhood W_E of U_E^c and increasing total pairwise soft neighbourhood V_E of x. So there is a pairwise open soft set G_E such that $U_E^c \sqsubseteq G_E \sqsubseteq W_E$. Since $V_E \sqsubseteq W_E^c$, then $V_E \sqsubseteq W_E^c \sqsubseteq G_E^c \sqsubseteq U_E$ and since G_E^c is soft closed, then $cl^s_{12}(V_E) \sqsubseteq G_E^c \sqsubseteq U_E$.

Sufficiency: Let x ∈ X and H_E be a decreasing pairwise closed soft set such that x 6∈ H_E. Then H_E^c be an increasing pairwise open soft set containing x. So that, by hypothesis, there is an increasing total pairwise soft neighbourhood V_E of x such that $cl_{12}^s(V_E) \sqsubseteq H_E^c$. Consequently, $(cl_{12}^s(V_E))^c$ is a pairwise open soft set containing H_E. Thus $d((cl_{12}^s(V_E))^c)$ is a decreasing pairwise soft neighbourhood of H_E. Suppose that $V_E \sqcap d((cl_{12}^s(V_E))^c) \neq \widehat{\phi}$.

Then there exists z ∈ X such that z ∈ V_E and $z \in d((cl_{12}^s(V_E))^c)$. So there exists $y \in ((cl_{12}^s(V_E))^c(e)$ satisfies that z ≤ y. This means that y ∈ V (e). But this contradicts the disjointedness between V_E and $(cl_{12}^s(V_E))^c$. Thus $V_E \sqcap d((cl_{12}^s(V_E))^c) = \widehat{\phi}$. This completes the proof.

A similar proof can be given for the case between parentheses.

Theorem 3.2 An SBTOS (X,τ_1,τ_2,E,≤) is lower (resp. upper) P*T−soft regularly ordered if and only if for all x ∈ X and every increasing (resp. decreasing) pairwise open soft set U_E containing x, there is an increasing (resp. decreasing) total pairwise soft neighbourhood V_E of x satisfies that $cl_{12}^s(V_E) \sqsubseteq U_E$.

Proof.The proof is similar to the proof of Theorem 3.1.

Theorem 3.3 An SBTOS(X,τ_1,τ_2,E,≤) is lower (resp. upper) PT (PP,P*P)−soft regularly ordered if and only if for all x ∈ X and every increasing (resp. decreasing) pairwise open soft set U_E containing x, there is an increasing (resp. decreasing) partial pairwise soft neighbourhood V_E of x satisfies that $cl_{12}^s(V_E) \sqsubseteq U_E$.

Proof.The proof is similar to the proof of Theorem 3.1.

Proposition 3.1 Every TP− soft T_3−ordered space (X,τ_1,τ_2,E,≤) is PP− soft T_3−ordered.

Proof. The proposition's proof establishes that the belong relation, denoted by ∈, can be extended to a partial belong relation denoted by b.

Example 3.1 Let E = {e_α,e_β,e_γ} be a set of parameters, ≤= N ∪ {(1,2)} be a partial order relation on the set of natural numbers ℵ. Define τ_1 = {G_E v $ℵ_E$ such that 1 6b G_E or [1 ∈ G(e_β) and G^c_E is finite]} and τ_2 = {F_E v $ℵ_E$ such that 3 ∈ F(e_α) and F_E^c is finite}. Obviously, (ℵ,τ_1,τ_2,E,≤) is PP− soft T_3−ordered [3]. A soft subset H_E of (ℵ,τ_1,τ_2,E,≤) is a decreasing pairwise closed soft set if [1 ∈ H_E and H_E is infinite] or [1 6∈ H(e_β), 3 6∈ H(e_α) and H_E is finite]. To illustrate that (ℵ,τ_1,τ_2,E,≤) is not lower PT−soft regularly ordered, we define a decreasing soft closed set H_E as follows: H_E = {(e_α,{1,2}),(e_β,{3}),(e_γ,{1,2})}.

Since 1 6∈ H_E and there do not exist disjoint soft neighbourhoods W_E and V_E containing H_E and 1, respectively, then (ℵ,τ_1,τ_2,E,≤) is not lower PT−soft regularly ordered. Hence (ℵ,τ_1,τ_2,E,≤) is not TP− soft T_3−ordered.

Proposition 3.2 An SBTOS (X,τ_1,τ_2,E,≤) is TP− soft T_3−ordered if and only if TP*− soft T_3−ordered.

Proof. On the one hand, x 6b G_E implies that x 6∈ G_E, then TP− soft T_3−ordered implies PP− soft T_3−ordered. On the other hand, the definition of PP− soft T_3−ordered implies that for every decreasing (resp. increasing) pairwise closed soft set H_E and x ∈ X such that x 6∈ H_E, there exist disjoint pairwise soft neighbourhood W_E of H_E and increasing (resp. decreasing) total pairwise soft neighbourhood V_E of x, such that W_E is decreasing (resp. increasing). Since W_E and V_E are disjoint, then x 6b H_E and ∀x 6≤ y, there exist an ITPS N_E of x such that y 6b N_E. Hence the definitions of TP− soft T_3−ordered and PP− soft T_3−ordered are equivalent.

Corollary 3.1 An SBTOS (X,τ_1,τ_2,E,≤) is PP− soft T_3−ordered if and only if PP*− soft T_3−ordered.

Proposition 3.3 Every TP*− soft T_3−ordered space (X,τ_1,τ_2,E,≤) is PP*− soft T_3−ordered.

Proof. The proof for the proposition states that the belong relation ∈ implies a partial belong relation b . The proposition is demonstrated by proving that the belong relation, represented by ∈, can be fully extended to a relation of partial belong, denoted by b.

Example 3.2 From Example 3.1, an SBTOS $(X,\tau_1,\tau_2,E,\leq)$ is PP^*- soft T_3-ordered but it is not TP^*- soft T_3-ordered.

Proposition 3.4 The following three properties are equivalent if $(X,\tau_1,\tau_2,E,\leq)$ is TP^*-soft regularly ordered:

1. $(X,\tau_1,\tau_2,E,\leq)$ is PST_2-ordered;
2. $(X,\tau_1,\tau_2,E,\leq)$ is PST_1-ordered;
3. $(X,\tau_1,\tau_2,E,\leq)$ is PST_0-ordered.

Proof. The direction 1) → 2) → 3) is obvious from Propositions 3.1,3.2 and 3.3 in [6].

To prove 3) → 1), let x,y ∈ X such that x 6≤ y. Since $(X,\tau_1,\tau_2,E,\leq)$ is PST_0- ordered, then it is lower pairwise soft T_1-ordered or upper pairwise soft T_1-ordered. Say it is upper pairwise soft T_1-ordered. From Theorem 3.1 in [6], we have that $(i(x))_E$ is pairwise soft closed. Obviously, $(i(x))_E$ is increasing and y 6b $(i(x))_E$. Since $(X,\tau_1,\tau_2,E,\leq)$ is TP^*- soft regularly ordered, then there exist disjoint decreasing total pairwise soft neighbourhood W_E of y and increasing pairwise soft neighbourhood V_E of $(i(x))_E$ so V_E is increasing total pairwise soft neighbourhood of x. Thus $(X,\tau_1,\tau_2,E,\leq)$ is PST_2-ordered.

Corollary 3.2 The following three properties are equivalent if $(X,\tau_1,\tau_2,E,\leq)$ is lower (upper) P^*T-soft regularly ordered:

1. $(X,\tau_1,\tau_2,E,\leq)$ is PST_2-ordered;
2. $(X,\tau_1,\tau_2,E,\leq)$ is PST_1-ordered;
3. $(X,\tau_1,\tau_2,E,\leq)$ is $LPST_1$(resp. $UPST_1$)−ordered.

Proposition 3.5 Every TP^*- soft T_3-ordered space $(X,\tau_1,\tau_2,E,\leq)$ is also PST_2-ordered.

Proof. Proposition 3.4 implies that any TP^*-soft T_3-ordered space is also PST_2-ordered.

Here is an illustration that shows that the converse of Proposition 3.5 is not necessarily true.

Example 3.3 Let E = $\{e_\alpha,e_\beta\}$ be a set of parameters, ≤= N ∪ {(1,2)} be a partial order relation on the set of natural numbers ℵ. Define $\tau_1 = \{G_E$ v $\aleph_E$ such that $1 \in G_E$ *and* G^c_E is infinite } and $\tau_2 = \{F_E$ v $\aleph_E$ such that $1 \in F^c_E\} \cup \aleph_E$. Then $(\aleph,\tau_1,\tau_2,E,\leq)$ is a soft bitopological ordered space. We have the following 6 cases:

For x,y ∈ N − {1,2}, x 6= y :

1. Either 1 6≤ x. Then we define two soft sets W_E and V_E as follows $W_E = \{(e_\alpha,\{1,2\}),(e_\beta,\{1,2\})\}$ and $V_E = \{(e_\alpha,\{x\}),(e_\beta,\{x\})\}$. So W_E is an increasing total pairwise soft neighbourhood of 1,V_E is a decreasing total pairwise soft neighbourhood of x and W_E u $V_E = \varphi$.b
2. Or x 6≤ 1. Then we define two soft sets W_E and V_E as follows $W_E = \{(e_\alpha,\{x\}),(e_\beta,\{x\})\}$ and $V_E = \{(e_\alpha,\{1\}),(e_\beta,\{1\})\}$. So W_E is an increasing total pairwise soft neighbourhood of x,V_E is a decreasing total pairwise soft neighbourhood of 1 and W_E u $V_E = \varphi$.b
3. Or 2 6≤ x. Then we define two soft sets W_E and V_E as follows $W_E = \{(e_\alpha,\{2\}),(e_\beta,\{2\})\}$ and $V_E = \{(e_\alpha,\{x\}),(e_\beta,\{x\})\}$. So W_E is an increasing total pairwise soft neighbourhood of 2,V_E is a decreasing total pairwise soft neighbourhood of x and W_E u $V_E = \varphi$.b
4. Or 2 6≤ 1. Then we define two soft sets W_E and V_E as follows $W_E = \{(e_\alpha,\{2\}),(e_\beta,\{2\})\}$ and $V_E = \{(e_\alpha,\{1\}),(e_\beta,\{1\})\}$. So W_E is an increasing total pairwise soft neighbourhood of 2,V_E is a decreasing total pairwise soft neighbourhood of 1 and W_E u $V_E = \varphi$.b
5. Or x 6≤ 2. Then we define two soft sets W_E and V_E as follows $W_E = \{(e_\alpha,\{x\}),(e_\beta,\{x\})\}$ and $V_E = \{(e_\alpha,\{1,2\}),(e_\beta,\{1,2\})\}$. So W_E is an increasing total pairwise soft neighbourhood of

x,V_E is a decreasing total pairwise soft neighbourhood of 2 and W_E u V_E = φ.b

6. Or x 6≤ y. Then we define two soft sets W_E and V_E as follows $W_E = \{(e_\alpha,\{x\}),(e_\beta,\{x\})\}$ and $V_E = \{(e_\alpha,\{y\}),(e_\beta,\{y\})\}$. So W_E is an increasing total pairwise soft neighbourhood of x,V_E is a decreasing total pairwise soft neighbourhood of y and W_E u V_E = φ.b

To illustrate that (ℵ,τ_1,τ_2,E,≤) is not lower PT−soft regularly ordered, we define a decreasing pairwise soft set H_E as follows: $H_E = \{(e_\alpha,\{1,2,4,5,...\}),(e_\beta,\{1,2,4,5,...\})\}$. Since 3 ∈6 H_E and there do not exit disjoint pairwise soft neighbourhoods W_E and V_E of H_E and 3, respectively, then (ℵ,τ_1,τ_2,E,≤) is not lower PT−soft regularly ordered.

Definition 3.3 An SBTOS (X,τ_1,τ_2,E,≤) is said to be:

1. Soft pairwise normally ordered if for each disjoint pairwise closed soft sets F_E and H_E such that F_E is increasing and H_E is decreasing, there exist disjoint pairwise soft neighborhoods W_E of F_E and V_E of H_E such that W_E is increasing and V_E is decreasing.
2. TP−soft T_4−ordered if it is soft pairwise normally ordered and $PST_1^{\bullet}-$ ordered.

Theorem 3.4 An SBTOS (X,τ_1,τ_2,E,≤) is soft pairwise normally ordered if and only if for every decreasing (resp. increasing) pairwise closed soft set F_E and every decreasing (resp. increasing) pairwise soft neighbourhood U_E of F_E, there is a decreasing (resp. increasing) pairwise soft neighbourhood V_E of F_E, satisfies that $cl^s_{12}(V_E) \sqsubseteq U_E$.

Proof. Necessity: Let F_E be a decreasing pairwise closed soft set and U_E be a decreasing pairwise soft neighbourhood of F_E. Then, U_E^c is an increasing pairwise closed soft set and F_E u U_E^c = φ.b Since (X,τ_1,τ_2,E,≤) is soft pairwise normally ordered, then there exist disjoint decreasing pairwise soft neighbourhood V_E of F_E and increasing pairwise soft neighbourhood W_E of U_E^c. Since W_E is a pairwise soft neighbourhood of U_E^c, then there exists a pairwise closed soft set H_E such that $U_E^c \sqsubseteq H_E \sqsubseteq W_E$. Consequently, $W_E^c \sqsubseteq H_E^c \sqsubseteq U_E$ and $V_E \sqsubseteq W_E^c$. So it follow that $cl^s_{12}(V_E) \sqsubseteq cl^s_{12}(W_E^c) \sqsubseteq H_E^c \sqsubseteq U_E$. Thus $F_E \sqsubseteq cl^s_{12}(V_E) \sqsubseteq cl^s_{12}(W_E^c) \sqsubseteq H_E^c \sqsubseteq U_E$. Hence the necessity part holds.

Sufficiency: Let F_E^1 and F_E^2 be two disjoint pairwise closed soft sets such that F_E^1 is decreasing and F_E^2 is increasing. Then F_E^{2c} is a decreasing pairwise open soft set containing F_E^1. By hypothesis, there exists a decreasing pairwise soft neighbourhood V_E of F_E^1 such that $cl^s_{12}(V_E) \sqsubseteq F_E^{2c}$. Setting $H_E = X_E - cl^s_{12}(V_E)$. $\widehat{This}$ means that H_E is a pairwise open soft set containing F_E^2. Obviously, F_E^2 v H_E, F_E^1 v V_E and H_E u V_E = φ. Now, i(H_E) is an increasing pairwise soft neighbourhood of F_E^2. Suppose that i(H_E)uV_E 6= φ.b Then there exists e ∈ E and x ∈ X such that x ∈ i(H_E) and x ∈ V (e) = d(V (e)). This implies that there exist a ∈ H(e) and b ∈ V (e) such that a ≤ x and x ≤ b. As ≤ is transitive, then a ≤ b. Therefore b b H_E u V_E. This contradicts the disjointness between H_E and V_E. Thus i(H_E) u V_E = φ.b Hence the proof is completed.

A proof similar can be given for the statement inside the parentheses.

Proposition 3.6 Every TP− soft T_4−ordered space (X,τ_1,τ_2,E,≤) is also TP*− soft T_3−ordered.

Proof. Let a ∈ X and F_E be a decreasing pairwise closed soft set such that a b F_E. Since (X,τ_1,τ_2,E,≤) is PST•−ordered, then $(i(a))_E$ is an increasing pairwise closed soft set and since (X,τ_1,τ_2,E,≤) is soft pairwise normally ordered, then there exist disjoint pairwise soft neighbourhood W_E and V_E of $(i(a))_E$ and F_E respectively, such that W_E is increasing and V_E is decreasing. Therefore, (X,τ_1,τ_2,E,≤) is lower PT−soft regularly ordered. If F_E is an increasing pairwise soft set, then we prove similarly that (X,τ_1,τ_2,E,≤) is upper

PT−soft regularly ordered. Thus $(X,\tau_1,\tau_2,E,\leq)$ is TP*− soft regularly ordered. Hence $(X,\tau_1,\tau_2,E,\leq)$ is TP*− soft T_3−ordered. The converse of the above proposition is not always true as illustrated in the following example.

Example 3.4 From Example (4.28) in [3], if we take $\tau_1 = \{G_E$ v $\aleph_E$ such that 1 b $G_E\}$ and $\tau_2 = \{F_E$ v $\aleph_E$ such that $1 \in F(e_2)$ and F_E^c is finite }. Then we have $(\aleph,\tau_1,\tau_2,E,\leq)$ is TP*− soft T_3−ordered, but it is not TP−soft T_4−ordered.

Theorem 3.5 The property of being a $PST_i\ (PST_i^{\bullet}, PST_i^{*}, PST_i^{**})-$ ordered space is soft bitopological ordered property, for i = 0,1,2.

Proof. We prove the theorem in case of PST_2 and the other follow similar lines.

Suppose that φ_ψ is an ordered embedding soft homeomorphism map of a PST_2−ordered space $(X,\tau_1,\tau_2,E,\leq_1)$ on to an SBTOS $(Y,\eta_1,\eta_2,K,\leq_2)$ and let $x,y \in Y$ such that x $\leq 6_2$ y. Then $x^e \leq 6_2\, y^e$, $\forall e \in E$. Since φ_ψ is bijective, then there exist a^α and b^α in X_E such that $\varphi_\psi(a^\alpha) = x^e$ and $\varphi_\psi(b^\alpha) = y^e$ and since φ_ψ is ordered embedding, then $a^\alpha\ 6\leq_1 b^\alpha$. So a $6\leq_1$ b. By hypothesis, there exist disjoint pairwise soft neighbourhoods W_E and V_E of a and b, respectively, such that W_E is increasing and V_E is decreasing. Since φ_ψ is bijective soft open, then $\varphi_\psi(W_E)$ and $\varphi_\psi(V_E)$ are disjoint soft neighbourhoods of x and y, respectively. It follows by Theorem 2.2, that $\varphi_\psi(W_E)$ is increasing and $\varphi_\psi(V_E)$ is decreasing. This completes the proof.

Theorem 3.6 The property of being a TP*(PP*)−soft T_3−ordered space is soft bitopological ordered property.

Proof. The proof is similar to the previous theorem.

Theorem 3.7 The property of being a TP−soft T_4−ordered space is soft bitopological ordered property.

Proof. Suppose that φ_ψ is an ordered embedding soft homeomorphism map of a soft pairwise normally ordered space $(X,\tau_1,\tau_2,E,\leq_1)$ on to an SBTOS $(Y,\eta_1,\eta_2,K,\leq_2)$ and let H_E and F_E be two disjoint pairwise closed soft sets such that H_E is increasing and F_E is decreasing. Since φ_ψ is bijective soft continuous, then $\phi_\psi^{-1}(H_E)$ and $\phi_\psi^{-1}(F_E)$ are disjoint pairwise closed soft sets and since φ_ψ is ordered embedding, then $\phi_\psi^{-1}(H_E)$ is increasing and $\varphi^-{}_\psi{}^1(F_E)$ is decreasing. By hypothesis, there exist disjoint pairwise soft neighbourhoods W_E and V_E of $\varphi^-{}_\psi{}^1(H_E)$ and $\varphi^-{}_\psi{}^1(F_E)$, respectively, such that W_E is increasing and V_E is decreasing. So H_E v $\varphi_\psi(W_E)$ and F_E v $\varphi_\psi(V_E)$. The disjointness of the soft neighbourhoods $\varphi_\psi(W_E)$ and $\varphi_\psi(V_E)$ completes the proof.

4. Conclusion

This research introduces a new concept called ”Bi-ordered soft regularity and normality” and investigates its properties. The study considers the notions of belong, non-belong, partial belong, and total non-belong in order to understand their interrelationships. Several examples are presented to facilitate comprehension.

In future research, the goal is to explore novel bi-ordered soft separation axioms by leveraging these concepts on supra soft topological spaces. The hope is that this work will inspire further research and lead to advancements in the field of soft topology.

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