New Elzaki Variational Iteration Method of Nonlinear Partial Differential Equations

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Abstract
In this paper we give a good strategy for solving some linear and nonlinear partial differential equations in applied mathematics fields, by combining Elzaki transform and the modified variational iteration method. This method is based on the variational iteration method, Elzaki transforms and convolution integral, such that, we introduce an alternative Elzaki correction functional and expressing the integral as a convolution. Some examples in applied mathematics are provided to illustrate the simplicity and reliability of this method. The solutions of these examples are contingent only on the initial conditions.
Keywords:
Elzaki transform, He's a variational iteration method, Nonlinear partial differential equations, Convolution integral.

1. Introduction

Nonlinear equations are of great importance to our contemporary world. Nonlinear phenomena have important applications in applied mathematics, physics, and issues related to engineering. Despite the importance of obtaining the exact solution of nonlinear partial differential equations in physics and applied mathematics there is still the daunting problem of finding new methods to discover new exact or approximate solutions.

In the recent years, many authors have devoted their attention to study solutions of nonlinear partial differential equations using various methods. Among these attempts are the Adomian decomposition method, homotopy perturbation method, variational iteration method [1-5], Laplace variational iteration method [6-8] differential transform method, Elzaki transform[14-17] and projected differential transform method.
Many analytical and numerical methods have been proposed to obtain solutions for nonlinear PDEs with fractional derivatives, such as local fractional variational iteration method [9], local fractional Fourier method, Yang-Fourier transform and Yang-Laplace transform and other methods. Two Laplace variational iteration methods are currently suggested by Wu in [10-13].
In this paper, we will introduce the new method termed He's a variational iteration method, and it will be employed in a straight forward manner.
Also, the main result of this paper is to introduce an alternative Elzaki correction functional and expressing the integral as a convolution. This approach cans tackle functions with discontinuities as well as impulse functions effectively. ELzaki transform, henceforth designated by the operator $\mathrm{E}[.]$, is defined by the integral equation.

$$\mathrm{E}\left[f(t)\right]=T(v)= v^2\int_0^\infty f(vt)\,e^{-t}dt ,$$

Theorem 1[17]:

Let $T'(v)$ be the ELzaki transforms of the derivative of $f(t)$. Then:

$$(i)\ T'(v)=\frac{T(v)}{v}-v\,f(0)$$

$$(ii)\ T^{(n)}(v)=\frac{T(v)}{v^n}-\sum_{k=0}^{n-1}v^{2-n+k}f^{(k)}(0) \qquad , \quad n\geq 1$$

Where $T^n(v)$ is ELzaki transform of the nth derivative of the function $f(t)$.

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Theorem 2 (convolution) [17]:

Let $f(t)$ *and* $g(t)$ having ELzaki transforms $M(v)$ and $N(v)$, then ELzaki transform of the Convolution of f and g ,

$$(f*g)(t)=\int_0^{\infty} f(t)g(t-\tau)\,d\tau, \text{ is given by: } \quad E\left[(f*g)(t)\right]=\frac{1}{v}M(v)N(v)$$

2. New Elzaki Variational Iteration Method:

To illustrate the idea of new Elzaki variational iteration method, we consider the following general differential equations in physics:

$$L\left[u(x,t)\right]+N\left[u(x,t)\right]=h(x,t) \tag{1}$$

Where L is a linear partial differential operator given by $\frac{\partial^2}{\partial t^2}$, N is nonlinear operator and $h(x,t)$ is a known analytical function.

According to the variational iteration method, we can construct a correction function for equation (1) as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \bar{\lambda}(x,\varsigma)\left[Lu_n(x,\varsigma)+N\tilde{u}_n(x,\varsigma)-h(x,\varsigma)\right]d\varsigma\ ,\ n\geq 0 \tag{2}$$

Where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the subscripts n denote the nth approximation, $N\tilde{u}_n(x,\varsigma)$ is considered as a restricted variation, i.e. $\delta N\tilde{u}_n(x,\varsigma)=0$.

Also we can find the Lagrange multipliers, by using integration by parts of Eq. (1), but in this paper, the Lagrange multipliers are found to be of the form $\lambda=\bar{\lambda}(x,t-\varsigma)$, and in such a case, the integration is basically the single convolution with respect to t , hence Elzaki transform is appropriate to use

Take Elzaki transform of eq. (2), then the correction functional will be constructed in the form

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+E\left[\int_0^t \bar{\lambda}(x,\varsigma)\left[Lu_n(x,\varsigma)+N\tilde{u}_n(x,\varsigma)-h(x,\varsigma)\right]d\varsigma\right],\ n\geq 0 \tag{3}$$

Therefore

$$\begin{aligned} E\left[u_{n+1}(x,t)\right]&=E\left[u_n(x,t)\right]+E\left[\bar{\lambda}(x,t)*\left[Lu_n(x,t)+N\tilde{u}_n(x,t)-h(x,t)\right]\right] \\ &=E\left[u_n(x,t)\right]+\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]E\left[Lu_n(x,t)+N\tilde{u}_n(x,t)-h(x,t)\right] \end{aligned} \tag{4}$$

Where * is a single convolution with respect to t .

To find the optimal value of $\bar{\lambda}(x,t-\varsigma)$ we first take the variation with respect to $u_n(x,t)$. Thus,

$$\frac{\delta}{\delta u_n}E\left[u_{n+1}(x,t)\right]=\frac{\delta}{\delta u_n}E\left[u_n(x,t)\right]+\frac{\delta}{\delta u_n}\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]E\left[Lu_n(x,t)+N\tilde{u}_n(x,t)-h(x,t)\right] \tag{5}$$

Then eq (5) becomes

$$E\left[\delta u_{n+1}(x,t)\right]=E\left[\delta u_n(x,t)\right]+\delta\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]E\left[Lu_n(x,t)\right] \tag{6}$$

In this paper we assume that L is a linear partial differential operator given by $\frac{\partial^2}{\partial t^2}$, then eq (6) can be written in the form:

$$E\left[\delta u_{n+1}(x,t)\right]=E\left[\delta u_n(x,t)\right]+E\left[\bar{\lambda}(x,t)\right]\left[\frac{1}{v^3}\delta u_n(x,t)\right] \tag{7}$$

The extreme condition of $u_{n+1}(x,t)$ requires that $\delta u_{n+1}(x,t)=0$. This means that the right hand side of eq (7) should be set to zero, then we have the following condition

$$E\left[\bar{\lambda}(x,t)\right]=-v^3 \;\Rightarrow\; \bar{\lambda}(x,t)=-t \tag{8}$$

Then we have the following iteration formula

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]-E\left[\int_0^t (t-\varsigma)\left[Lu_n(x,\varsigma)+N\tilde{u}_n(x,\varsigma)-h(x,\varsigma)\right]d\varsigma\right],\ n\geq 0 \tag{9}$$

3. Applications:

In this section we apply the Elzaki variational iteration method to solving some linear and nonlinear partial differential equations in physics.

Example 1:

Consider the initial linear partial differential equation

$$u_{tt}(x,t)-u_{xx}(x,t)+u(x,t)=0 \quad , \quad u(x,0)=0 \;\; , \;\; \frac{\partial u(x,0)}{\partial t}=x \tag{10}$$

The Elzaki variational iteration correction functional will be constructed in the following manner

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+E\left[\int_0^t \bar{\lambda}(x,t-\varsigma)\left[(u_n)_{tt}(x,\varsigma)-(u_n)_{xx}(x,\varsigma)+u_n(x,\varsigma)\right]d\varsigma\right] \tag{11}$$

Or

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+E\left[\bar{\lambda}(x,t)*\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n(x,t)\right]\right]$$

$$=E\left[u_n(x,t)\right]+\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]E\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n(x,t)\right] \tag{12}$$

$$=E\left[u_n(x,t)\right]+\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]\left[\begin{array}{l}\frac{1}{v^2}E(u_n(x,t))-u_n(x,0)-v\frac{\partial u_n}{\partial t}(x,0)-E(u_n)_{xx}(x,t)\\ +Eu_n(x,t)\end{array}\right]$$

Taking the variation with respect to $u_n(x,t)$ of eq (12), to obtain

$$\frac{\delta}{\delta u_n}E\left[u_{n+1}(x,t)\right]=\frac{\delta}{\delta u_n}E\left[u_n(x,t)\right]$$

$$+\frac{\delta}{\delta u_n}\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]\left[\begin{array}{l}\frac{1}{v^2}E(u_n(x,t))-u_n(x,0)-v\frac{\partial u_n}{\partial t}(x,0)-E(u_n)_{xx}(x,t)\\ +Eu_n(x,t)\end{array}\right] \tag{13}$$

Then we have

$$E\left[\delta u_{n+1}(x,t)\right]=E\left[\delta u_n(x,t)\right]+E\left[\bar{\lambda}(x,t)\right]\left[\frac{1}{v^3}E\,\delta u_n(x,t)+\frac{1}{v}E\,\delta u_n(x,t)\right]$$
$$=E\left[\delta u_n(x,t)\right]\left\{1+E\left[\bar{\lambda}(x,t)\right]\left(\frac{1}{v^3}+\frac{1}{v}\right)\right\}$$

The extreme condition of $u_{n+1}(x,t)$ requires that $\delta u_{n+1}(x,t)=0$. Hence, we have

$$1+\left(\frac{1}{v^3}+\frac{1}{v}\right)E\,\bar{\lambda}(x,t)=0 \quad , \quad and \quad \bar{\lambda}(x,t)=E^{-1}\left[\frac{-v^3}{v^2+1}\right]=-\sin t \tag{14}$$

Substituting eq (14) into eq (11), to obtain

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]-E\left[\int_0^t \sin(t-\varsigma)\left[(u_n)_{tt}(x,\varsigma)-(u_n)_{xx}(x,\varsigma)+u_n(x,\varsigma)\right]d\varsigma\right]$$
$$=E\left[u_n(x,t)\right]-\frac{1}{v}E\left[\sin t\right]E\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n(x,t)\right] \tag{15}$$

Let $u_0(x,t)=u(x,0)+t\,\frac{\partial u}{\partial t}(x,0)=xt$, then from eq (15), we have

$$E\left[u_1(x,t)\right]=E\left[xt\right]-\frac{1}{v}E\left[\sin t\right]E\left[xt\right]$$

The inverse Elzaki transforms yields

$$u_1(x,t)=xt-E^{-1}\left[xv^3-\frac{xv^3}{1+v^2}\right]=x\,\sin t \tag{16}$$

Substituting eq (16) into eq (11), to obtain

$$E\left[u_2(x,t)\right]=E\left[x\,\sin t\right]-E\left[\sin t\right]E\left[0\right] \qquad then \qquad u_2(x,t)=x\,\sin t$$

And then the exact solution of eq. (10) is

$$u(x,t)=x\,\sin t \tag{17}$$

We see that the exact solution is coming very fast by using only few terms of the iterative scheme.

Example 2:

Consider the nonlinear partial differential equation:

$$u_{tt}(x,t)-u_{xx}(x,t)+u^2(x,t)=x^2t^2 \quad , \quad u(x,0)=0 \quad , \quad \frac{\partial u(x,0)}{\partial t}=x \tag{18}$$

The Elzaki variational iteration correction functional will be constructed as follows

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+E\left[\int_0^t \bar{\lambda}(x,t-\varsigma)\begin{bmatrix}(u_n)_{tt}(x,\varsigma)-(u_n)_{xx}(x,\varsigma)\\ +u_n^2(x,\varsigma)-x^2t^2\end{bmatrix}d\varsigma\right] \tag{19}$$

Or

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+E\left[\bar{\lambda}(x,t)*\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n^2(x,t)-x^2t^2\right]\right]$$

$$=E\left[u_n(x,t)\right]+\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]E\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n^2(x,t)-x^2t^2\right] \quad (20)$$

$$=E\left[u_n(x,t)\right]+\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]\begin{bmatrix}\frac{1}{v^2}E(u_n(x,t))-u_n(x,0)-v\frac{\partial u_n}{\partial t}(x,0)-E(u_n)_{xx}(x,t)\\ +Eu_n^2(x,t)-E(x^2t^2)\end{bmatrix}$$

Taking the variation with respect to $u_n(x,t)$ of eq (20), and making the correction functional stationary to obtain

$$E\left[\delta u_{n+1}(x,t)\right]=E\left[\delta u_n(x,t)\right]+E\left[\bar{\lambda}(x,t)\right]\left[\frac{1}{v^3}\delta u_n(x,t)\right]$$

$$=E\left[\delta u_n(x,t)\right]\left\{1+\frac{1}{v^3}E\left[\bar{\lambda}(x,t)\right]\right\}$$

This implies that:

$$1+\frac{1}{v^3}E\bar{\lambda}(x,t)=0 \;,\; \text{and} \quad \bar{\lambda}(x,t)=E^{-1}\left[-\frac{1}{v^3}\right]=-t \quad (21)$$

Substituting eq (21) into eq (19), to obtain

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]-E\left[\int_0^t(t-\varsigma)\begin{bmatrix}(u_n)_{tt}(x,\varsigma)-(u_n)_{xx}(x,\varsigma)\\ +u_n^2(x,\varsigma)-x^2\varsigma^2\end{bmatrix}d\varsigma\right] \quad (22)$$

Or

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+\frac{1}{v}E\left[-t\right]E\left[(u_n)_{tt}(x,t)-(u_n)_{xx}(x,t)+u_n^2(x,t)-x^2t^2\right] \quad (23)$$

Let $u_0(x,t)=u(x,0)+t\frac{\partial u}{\partial t}(x,0)=xt$, then from eq (23), we have

$$E\left[u_1(x,t)\right]=E\left[xt\right]+\frac{1}{v}E\left[-t\right]E\left[0-0+x^2t^2-x^2t^2\right]$$

$$u_1(x,t)=xt$$

Then the exact solution of eq (18) is $u(x,t)=xt$

Again the exact solution is coming very fast by using only few terms of the iterative scheme.

Example 3:

Consider the physics nonlinear boundary value problem

$$u_t-6uu_x+u_{xxx}=0 \quad , \quad u(x,0)=\frac{6}{x^2} \;, \quad x\neq 0 \quad (24)$$

The Elzaki variational iteration correction functional is

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+E\left[\int_0^t\bar{\lambda}(x,t-\varsigma)\begin{bmatrix}(u_n)_t(x,\varsigma)-6u_n(x,\varsigma)(u_n)_x(x,\varsigma)\\ +(u_n)_{xxx}(x,\varsigma)\end{bmatrix}d\varsigma\right] \quad (25)$$

Or

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+E\left[\bar{\lambda}(x,t)*\left[(u_n)_t(x,t)-6(u_n)(x,t)(u_n)_x(x,t)+(u_n)_{xxx}(x,t)\right]\right]$$
$$=E\left[u_n(x,t)\right]+\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]E\left[(u_n)_t(x,t)-6(u_n)(x,t)(u_n)_x(x,t)+(u_n)_{xxx}(x,t)\right]$$
$$=E\left[u_n(x,t)\right]+\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]\left[\frac{1}{v}Eu_n(x,t)-vu_n(x,0)-E\left[6(u_n)(x,t)(u_n)_x(x,t)-(u_n)_{xxx}(x,t)\right]\right]$$

Taking the variation with respect to $u_n(x,t)$ of the last equation, and making the correction functional stationary to obtain

$$E\left[\delta u_{n+1}(x,t)\right]=E\left[\delta u_n(x,t)\right]+\frac{1}{v}E\left[\bar{\lambda}(x,t)\right]\left[\frac{1}{v}E\,\delta u_n(x,t)\right]$$
$$=E\left[\delta u_n(x,t)\right]\left\{1+\frac{1}{v^2}E\left[\bar{\lambda}(x,t)\right]\right\}$$

This implies that:

$$1+\frac{1}{v^2}E\,\bar{\lambda}(x,t)=0\ ,\quad and\quad \bar{\lambda}(x,t)=E^{-1}\left[-v^2\right]=-1 \tag{26}$$

Substituting eq (26) into eq (25), to obtain

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n(x,t)\right]+E\left[\int_0^t(-1)\begin{bmatrix}(u_n)_t(x,\varsigma)-6(u_n)(x,\varsigma)(u_n)_x(x,\varsigma)\\+(u_n)_{xxx}(x,\varsigma)\end{bmatrix}d\varsigma\right]$$

Or

$$E\left[u_{n+1}(x,t)\right]=E\left[u_n\right]+\frac{1}{v}E\left[-1\right]E\left[(u_n)_t-6(u_n)(u_n)_x+(u_n)_{xxx}\right] \tag{27}$$

Let $u_0(x,t)=u(x,0)=\dfrac{6}{x^2}$, then from eq (27), we have

$$E\left[u_1(x,t)\right]=E\left[\frac{6}{x^2}\right]+\frac{1}{v}E\left[-1\right]E\left[\frac{288}{x^5}\right]\quad\Rightarrow\quad u_1(x,t)=\frac{6}{x^2}-\frac{288}{x^5}t$$
$$u_2(x,t)=\frac{6}{x^2}-\frac{288}{x^5}t-\frac{6048}{x^8}t^2\ ,\ldots$$

Then the exact solution of eq (24) is $u(x,t)=\dfrac{6x\left(x^3-24t\right)}{\left(x^3-12t\right)^2}$

4. Conclusion:

The method of combining Elzaki transforms and variational iteration method is proposed for the solution of linear and nonlinear partial differential equations. This method is applied in a direct way without employing linearization and is successfully implemented by using the initial conditions and convolution integral.

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Appendix:

Elzaki transform of some functions

$f(t)$	$E[f(t)] = T(u)$
1	v^2
t	v^3
t^n	$n!\,v^{n+2}$
e^{at}	$\frac{v^2}{1-av}$
$sinat$	$\frac{av^3}{1+a^2v^2}$
$cosat$	$\frac{v^2}{1+a^2v^2}$